MATHEMATICS

THE DRAG ON A VIBRATING AEROFOIL IN INCOMPRESSIBLE FLOW. II

BY

G. W. VELTKAMP

(Communicated by Prof. H. FREUDENTHAL at the meeting of December 28, 1957)

5. *The behaviour of Ω and Ω' near $z=-1$*

By an analysis analogous to that which led to (26), we have

$$(30)\quad \frac{d\Omega_1}{dz}=-\frac{1}{\pi i}2^{-\frac{1}{2}}(-z-1)^{-\frac{1}{2}}\Gamma_{01}-i\,V(-1)+o((z+1)^{\varepsilon_1}) \text{ as } z\to-1,$$

uniformly for $0\leqslant \arg(z+1)\leqslant 2\pi$. Here ε_1 is any number that satisfies $0<\varepsilon_1<\varepsilon$ and $(-z-1)^{-\frac{1}{2}}$ is by definition continuous and positive for z real and <-1. Since, clearly, $\Omega_2(z)=0((z+1)^{\frac{1}{2}})$ as $z\to-1$, we have from (29)

$$A\frac{d\Omega_2}{dz}=-\frac{1}{\pi i}2^{-\frac{1}{2}}(-z-1)^{-\frac{1}{2}}[2C(k)-1]\,\Gamma_{10}+0((z+1)^{\frac{1}{2}} \text{ as } z\to-1.$$

Consequently, we have

$$(31)\quad \begin{cases}\dfrac{d\Omega}{dz}=-\dfrac{1}{\pi i}2^{-\frac{1}{2}}(-z-1)^{-\frac{1}{2}}\{\Gamma_{01}+[2C(k)-1]\,\Gamma_{10}\}+\\ \qquad -i\,V(-1)+o((z+1)^{\varepsilon_1}) \text{ as } z\to-1.\end{cases}$$

Finally, since $W(s)=0(s+1)$ as $s\to-1$, the integral in (18) is bounded as $z\to-1$, so $\Omega_1(z)=0((z+1)^{\frac{1}{2}})$ and also $\Omega(z)=0((z+1)^{\frac{1}{2}})$ as $z\to-1$. From these estimations it follows that condition (v) is satisfied.

6. *The behaviour of Ω and Ω' near $z=\infty$*

The function $\Omega_1(z)$ is regular at $z=\infty$ and consequently it admits a convergent expansion in powers of z^{-1}. From (18) (or, more simply, from (20)) the first terms of this expansion are easily found, viz:

$$(32)\qquad \Omega_1(z)=\text{const.}+\frac{\Gamma_{11}}{\pi iz}+\frac{\Gamma_{21}-\Gamma_{12}}{4\pi iz^2}+0(z^{-3}).$$

The behaviour of Ω_2 and Ω_2' at infinity can be found in the following way. For $0\leqslant\arg z\leqslant\frac{3\pi}{2}$ we shift the path of integration so, that it ultimately tends to infinity along the ray $\arg s=\frac{3\pi}{2}+\delta$, where δ is a small positive quantity. And for $\frac{3\pi}{2}\leqslant\arg z\leqslant 2\pi$ we let the path go to infinity along the ray $\arg s=\frac{3\pi}{2}-\delta$. In the latter case the pole $s=z$ of the integrand is passed (at least if $|z|$ is sufficiently large) and its residue has to be taken into account. In this way we find

$$\text{for} \quad 0 \leqslant \arg z \leqslant \frac{3\pi}{2} \quad \Omega_2(z) = \Phi^+(z),$$

$$\text{for} \ \frac{3\pi}{2} \leqslant \arg z \leqslant 2\pi \quad \Omega_2(z) = -e^{-ikz} + \Phi^-(z),$$

where

$$\Phi^{\pm}(z) = \frac{1}{2\pi i}(z^2-1)^{\frac{1}{2}} \int_1^{-i\infty \exp[\pm i\delta]} \frac{e^{-iks}}{(s^2-1)^{\frac{1}{2}}(s-z)} ds.$$

Since the integrand in the integrals for $\Phi^{\pm}$ decreases exponentially and $\arg z$ is bounded away from $\arg s$, it is easy to see that Φ^+ and Φ^- admit expansions in powers of z^{-1} that are asymptotically convergent, uniformly in $\arg z$ for $0 \leqslant \arg z \leqslant \frac{3\pi}{2}$ and $\frac{3\pi}{2} \leqslant \arg z \leqslant 2\pi$, respectively. The coefficients of these expansions are, of course, the same. Furthermore, term-by-term differentiation of these expansions gives asymptotic expansions for $\frac{d\Phi^+}{dz}$ and $\frac{d\Phi^-}{dz}$.

Consequently, we have, as $z \to \infty$, uniformly for $0 \leqslant \arg z \leqslant 2\pi$,

$$(33) \quad \begin{cases} A\Omega_2(z) = \begin{Bmatrix} 0 & \text{for } 0 \leqslant \arg z \leqslant \frac{3\pi}{2} \\ -Ae^{-ikz} & \text{for } \frac{3\pi}{2} \leqslant \arg z \leqslant 2\pi \end{Bmatrix} + \text{const.} + 0(z^{-1}), \\[2ex] A\frac{d\Omega_2}{dz} = \begin{Bmatrix} 0 & \text{for } 0 \leqslant \arg z \leqslant \frac{3\pi}{2} \\ ikAe^{-ikz} & \text{for } \frac{3\pi}{2} \leqslant \arg z \leqslant 2\pi \end{Bmatrix} + 0(z^{-2}) \end{cases}$$

and from (32) we infer that the same formulae hold for $\Omega(z)$ and $\frac{d\Omega}{dz}$, respectively. This shows that the regularity condition (vi) is satisfied.

Finally, from (32) and (29) we can derive the following expansion for the function

$$(34) \qquad P(z) \stackrel{\text{def}}{=} \frac{d\Omega}{dz} + ik\Omega$$

(that is regular at infinity)

$$(35) \quad \begin{cases} P(z) = \text{const.} + \frac{1}{\pi i z}[C(k)\,\Gamma_{10} + ik\,\Gamma_{11}] + \\ \qquad - \frac{1}{2\pi i z^2}\{2\Gamma_{11} - \frac{1}{2}ik(\Gamma_{21} - \Gamma_{12}) - [1-C(k)]\,\Gamma_{10}\} + 0(z^{-3}) \end{cases}$$

as $z \to \infty$.

7. *Velocity, pressure, lift, moment and wake*

From (4), (9), (10) and (34) we have

$$(36) \quad \begin{cases} u_1(x, y) = \frac{1}{2}[\Omega'(x+iy) - \Omega'(x-iy)], \\ v_1(x, y) = \frac{1}{2}i[\Omega'(x+iy) + \Omega'(x-iy)], \\ p_1(x, y) = -\frac{1}{2}[P(x+iy) - P(x-iy)], \end{cases}$$

so expressions for u_1, v_1 and p_1 are readily found from the general formulae for Ω and Ω'. At Λ_1 and Λ_2 these formulae will, in general, involve Cauchy principal-value integrals, since by Plemelj's formulae we have, e.g., at Λ_1

$$\Omega_1^+ - \Omega_1^- = i(1-x^2)^{\frac{1}{2}}(\chi^+ + \chi^-) = -\frac{2}{\pi}(1-x^2)^{\frac{1}{2}} \int_{-1}^{1} \frac{W(s)}{(1-s^2)^{\frac{1}{2}}(s-x)} ds.$$

In this way we find for the pressure at the aerofoil

$$p_1^+ = -p_1^- = \frac{1}{\pi}(1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} \int_{-1}^{1} \left[\frac{1}{s-x} - 1 + C(k)\right](1+s)^{\frac{1}{2}}(1-s)^{-\frac{1}{2}} V(s)\, ds +$$
$$+ \frac{ik}{\pi}(1-x^2)^{\frac{1}{2}} \int_{-1}^{1} \frac{W(s)}{(1-s^2)^{\frac{1}{2}}(s-x)} ds. \text{ [10]}$$

From this expression formulae for the lift L and the moment about the centre of the aerofoil M could be found by integration.
The following derivation is much easier, however. If

$L = \mathrm{Re}\,[L_1 e^{ikt}]$, $M = \mathrm{Re}\,[M_1 e^{ikt}]$, then we have from (36)

$$L_1 = \int_{-1}^{1} (p_1^- - p_1^+)\, dx = -\int_{-1}^{1} (P^- - P^+)\, dx.$$

Now $P(z)$ is regular outside Λ_1, $0((z+1)^{-\frac{1}{2}})$ as $z \to -1$ and $0(1)$ as $z \to 1$. Consequently, in virtue of Cauchy's theorem we have

$$L_1 = -\oint_C P(z)\, dz,$$

where C is any contour encircling Λ_1 in the positive direction. Letting C tend to infinity, we find from (35)

$$L_1 = -2[C(k)\,\Gamma_{10} + ik\,\Gamma_{11}] = -2\int_{-1}^{1} \left[\frac{C(k)}{1-s} + ik\right](1-s^2)^{\frac{1}{2}} V(s)\, ds.$$

Similarly,

$$M_1 = \int_{-1}^{1} x(p_1^- - p_1^+)\, dx = -\oint z P(z)\, dz =$$
$$= 2\Gamma_{11} - \tfrac{1}{2} ik(\Gamma_{21} - \Gamma_{12}) - [1 - C(k)]\,\Gamma_{10} =$$
$$= \int_{-1}^{1} \left[2 - iks - \frac{1-C(k)}{1-s}\right](1-s^2)^{\frac{1}{2}} V(s)\, ds.$$

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[10]) By integration by parts the last term can be transformed into

$$\frac{ik}{\pi}\int_{-1}^{1} \Lambda(x,s)\, V(s)\, ds,$$

where

$$\Lambda(x,s) = \tfrac{1}{2}\log\frac{1-xs+(1-x^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}}}{1-xs-(1-x^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}}}.$$

The character of the wake far downstream is directly found from (36) and (33). Putting $(x^2+y^2)^{\frac{1}{2}}=R$, we have, as $x\to+\infty$, uniformly in y,

$$(37)\qquad \begin{cases} \varphi_1=\frac{1}{2}\,A\ \operatorname{sgn}\ y\, e^{-ikx-k|y|}+O(R^{-1}),\\ u_1=-\frac{1}{2}ik\,A\ \operatorname{sgn} y\, e^{-ikx-k|y|}+O(R^{-2}),\\ v_1=-\frac{1}{2}k\,A\ e^{-ikx-k|y|}+O(R^{-2}),\\ p_1=O(R^{-1}).\end{cases}$$

Thus it is seen that the wake consists of a progressive wave moving with the velocity of the undisturbed stream. The amplitude decreases exponentially in the direction normal to the line of discontinuity.

8. *The drag*

The drag D on the aerofoil is the sum of the suction D' due to the singularity at the leading edge and the integral D'' of the horizontal components of the thrust on the remaining part of the aerofoil. We shall here evaluate the time averages of this quantities.

It follows from a consideration of the balance of linear momentum for a small circle C_ϱ with radius ϱ around the leading edge that

$$(38)\qquad D'=-\lim_{\varrho\to 0}\int_{C_\varrho} u\, w_n\, ds,$$

where w_n is the component of the velocity along the outward normal to C_ϱ. This expression is correct in second-order approximation [11]. According to (2), the time-average of D' is

$$[D']_{\mathrm{Av}}=-\tfrac{1}{2}\operatorname{Re}\ [\lim_{\varrho\to 0}\int_{C_\varrho} u_1\bar{w}_{1n}ds],$$

where $\bar{w}_{1n}$ is the complex conjugate of w_{1n}. From (31) we have, putting $z=-1-\varrho e^{i\vartheta}$, $-\pi\leqslant\vartheta\leqslant\pi$,

$$\frac{d\Omega}{dz}=-\frac{1}{\pi i}2^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}e^{-\frac{1}{2}i\vartheta}\{\Gamma_{01}+[2C(k)-1]\,\Gamma_{10}\}+0(1)$$

as $\varrho\to 0$, whence, by (36),

$$\begin{aligned} u_1\bar{w}_{1n}&=-u_1(\bar{u}_1\ \cos\ \vartheta+\bar{v}_1\ \sin\ \vartheta)=\\ &=\frac{1}{2\pi^2\varrho}\,|\,\Gamma_{01}+[2C(k)-1]\,\Gamma_{10}\,|^2\sin^2\tfrac{1}{2}\vartheta+0(\varrho^{-\frac{1}{2}})\end{aligned}$$

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[11]) At first sight it seems remarkable that a linearized theory could give a correct approximation of a second-order effect that is due to the flow in a region where the conditions allowing the linearization are certainly violated. However, as has been indicated by LIGHTHILL [14], the approximation obtained by the linearized theory for the velocity near a leading edge of great curvature can be made uniformly valid by multiplication with a factor depending only on the radius of curvature of the leading edge. And thus it is seen that (38) gives a result that is correct in second-order approximation.

as $\varrho \to 0$. Hence the average value of the suction is

$$[D']_{\mathrm{Av}} = -\frac{1}{4\pi} \,|\, \Gamma_{01} + [2C(k)-1]\; \Gamma_{10} \,|^2 = $$
$$= -\frac{1}{\pi} \,\Big|\, \int_{-1}^{1} \{C(k) - [1-C(k)]\, s\}\; (1-s^2)^{-\frac{1}{2}}\; V(s)\; ds \,\Big|^2.$$

The average value of the integral of the horizontal components of the thrust on the remaining part of the aerofoil is (in second-order approximation)

$$[D'']_{\mathrm{Av}} = \tfrac{1}{2}\,\mathrm{Re}\Big[\int_{-1}^{1} (p_1^+ - p_1^-)\, \frac{d\bar{f}_1}{dx}\, dx\Big] =$$
$$= -\tfrac{1}{2}\,\mathrm{Re}\Big[\int_{-1}^{1} (P^+ - P^-)\, \frac{d\bar{f}_1}{dx}\, dx\Big].$$

This quantity will be calculated in two parts by putting $P(z) = P_1(z) + + AP_2(z)$, where $P_j(z) = \frac{d\Omega_j}{dz} + ik\Omega_j$ $(j = 1, 2)$.

For the contribution I_1 of P_1 we have, integrating by parts and using the fact that $\Omega_1^+ - \Omega_1^- \to 0$ as $x \to \pm 1$,

$$I_1 = -\tfrac{1}{2}\,\mathrm{Re}\Big[\int_{-1}^{1} \frac{d\bar{f}_1}{dx}\Big(\frac{d}{dx} + ik\Big)(\Omega_1^+ - \Omega_1^-)\, dx\Big] =$$
$$= -\tfrac{1}{2}\,\mathrm{Re}\Big[\int_{-1}^{1} \Big(\frac{d\Omega_1{}^+}{dz} - \frac{d\Omega_1{}^-}{dz}\Big)\Big(\frac{d}{dx} - ik\Big)\bar{f}_1\, dx\Big].$$

Putting for brevity

$$G(z) \overset{\mathrm{def}}{=} \frac{d\Omega_1}{dz},\; H(z) \overset{\mathrm{def}}{=} \overline{G(\bar{z})},$$

we have at Λ_1

$$\bar{G}^+ = H^-,\;\; \bar{G}^- = H^+.$$

And from (6) and (11) we have, since

$$\frac{d\Omega_2{}^+}{dz} + \frac{d\Omega_2{}^-}{dz} = 0 \;\text{ at }\; \Lambda_1,$$

$$\Big(\frac{d}{dx} - ik\Big)\bar{f}_1 = \overline{V(x)} = -\tfrac{1}{2}\, i\, (H^+ + H^-). \tag{39}$$

Consequently

$$I_1 = \tfrac{1}{4}\,\mathrm{Re}\,[i \int_{-1}^{1} (G^+ - G^-)\,(H^+ + H^-)\, dx] =$$
$$= \tfrac{1}{8}\, i \int_{-1}^{1} [(G^+ - G^-)\,(H^+ + H^-) - (H^- - H^+)\,(G^- + G^+)]\; dx =$$
$$= \tfrac{1}{4}\, i \int_{-1}^{1} (G^+ H^+ - G^- H^-)\, dx.$$

This integral, the integrand of which is the difference of the boundary values of the sectionally holomorphic function $G(z)H(z)$ at its line of discontinuity, is easily evaluated. From (30) we have

$$G(z)\,H(z) = -\frac{|\Gamma_{01}|^2}{2\pi^2}\cdot\frac{1}{z+1} + 0((z+1)^{-\frac{1}{2}}) \text{ as } z \to -1$$

and from (26) we have

$$G(z)\,H(z) = \frac{|\Gamma_{10}|^2}{2\pi^2}\cdot\frac{1}{z-1} + 0((z-1)^{-\frac{1}{2}}) \text{ as } z \to +1.$$

Since $G(z)H(z) = 0(z^{-4})$ as $z \to \infty$, we have by an obvious application of Cauchy's theorem

$$\int_{-1}^{1} (G^- H^- - G^+ H^+)\,dx - \frac{i}{\pi}|\Gamma_{01}|^2 + \frac{i}{\pi}|\Gamma_{10}|^2 = 0,$$

whence

$$I_1 = \frac{1}{4\pi}\,[\,|\Gamma_{01}|^2 - |\Gamma_{10}|^2].$$

For the evaluation of the contribution I_2 of AP_2 we use (29), whence

$$I_2 = -\tfrac{1}{2}\,\mathrm{Re}\Big[\int_{-1}^{1} A\,(P_2^+ - P_2^-)\frac{d\overline{f_1}}{dx}\,dx\Big] =$$
$$= \frac{1}{\pi}\,\mathrm{Re}\,\Big[\Gamma_{10}\int_{-1}^{1}\{x+(1-x)\,C(k)\}\,(1-x^2)^{-\frac{1}{2}}\frac{d\overline{f_1}}{dx}\,dx\Big].$$

The average drag $[D]_{\mathrm{Av}}$ (including suction) is the sum of $[D']_{\mathrm{Av}'}$, I_1 and I_2:

$$[D]_{\mathrm{Av}} = -\frac{1}{4\pi}\,|\Gamma_{01} + [2\,C(k)-1]\,\Gamma_{10}|^2 + \frac{1}{4\pi}\,[\,|\Gamma_{01}|^2 - |\Gamma_{10}|^2] +$$
$$+ \frac{1}{\pi}\,\mathrm{Re}\,\Big[\Gamma_{10}\int_{-1}^{1}\{x+(1-x)\,C(k)\}\,(1-x^2)^{-\frac{1}{2}}\frac{d\overline{f_1}}{dx}\,dx\Big].$$

An alternative expression is found by substitution of $\frac{d\overline{f_1}}{dx} = \overline{V} + ik\overline{f_1}$ in the integral for I_2, which gives after some algebra

$$(40)\quad \left\{\begin{aligned} &[D]_{\mathrm{Av}} = \frac{1}{2\pi}\,[C(k) + \overline{C(k)} - 2\,C(k)\,\overline{C(k)}]\,|\,\Gamma_{10}|^2 + \\ &\qquad - \frac{k}{\pi}\,\mathrm{Im}\,\;[\Gamma_{10}\int_{-1}^{1}\{x+(1-x)\,C(k)\}\,(1-x^2)^{-\frac{1}{2}}\,\overline{f_1(x)}\,dx].\end{aligned}\right.$$

In the case that $f_1(x)$ is a polynomial, the expression for $[D]_{\mathrm{Av}}$ can be shown to be equivalent to that of Jones (l.c. p. 15).

9. *The balance of energy*

Let W_L and W_D be the work that is done in unit time on the aerofoil by the lift and the drag, respectively. Then $-W_L - W_D$ is the work done in unit time by the aerofoil on the fluid. Its time average $-[W_L]_{\mathrm{Av}}$ $-[W_D]_{\mathrm{Av}}$ should be equal to the average rate of kinetic energy $[W_W]_{\mathrm{Av}}$

that is carried off in the wake. We shall calculate these three quantities separately [12]).

Since the vertical velocity of the aerofoil is

$$\operatorname{Re}\,[i\,k\,f_1(x)\,e^{ikt}],$$

we have, using (2),

$$[W_L]_{\mathrm{Av}}=\tfrac{1}{2}\operatorname{Re}\,[\int_{-1}^{1}(p_1^- - p_1^+)\,(\overline{i\,k\,f_1})\,dx]=-\tfrac{1}{2}k\operatorname{Re}\,[i\int_{-1}^{1}(P^+ - P^-)\,\overline{f_1}\,dx],$$

which is correct in second-order approximation.

Let us consider the contributions of P_1 and AP_2 separately. By integration by parts and using (39) we have for the contribution of P_1

$$\begin{aligned}
I&=\tfrac{1}{2}k\operatorname{Re}\,[i\int_{-1}^{1}(P_1^+ - P_1^-)\,\overline{f_1}\,dx]=\\
&=\tfrac{1}{2}k\operatorname{Re}\Big[i\int_{-1}^{1}(\Omega_1^+ - \Omega_1^-)\Big(\frac{d}{dx}-ik\Big)\overline{f_1}\,dx\Big]=\\
&=\tfrac{1}{4}k\operatorname{Re}\,[\int_{-1}^{1}(\Omega_1^+ - \Omega_1^-)\,(H^+ + H^-)\,dx]=\\
&=\tfrac{1}{8}k\,[\int_{-1}^{1}(\Omega_1^+ - \Omega_1^-)\,(H^+ + H^-)\,dx+\int_{-1}^{1}(\overline{\Omega_1^+}-\overline{\Omega_1^-})\,(G^+ + G^-)\,dx].
\end{aligned}$$

Partial integration in the second term gives, since

$$(G^+ + G^-)=\frac{d}{dx}(\Omega_1^+ + \Omega_1^-),\ \frac{d}{dx}(\overline{\Omega_1^+}-\overline{\Omega_1^-})=H^- - H^+,$$

$$I=\tfrac{1}{4}k\int_{-1}^{1}(\Omega_1^+\,H^+ - \Omega_1^-\,H^-)\,dx.$$

Since $\Omega_1 H$ is $0(1)$ as $z\to -1$, $0((z-1)^{-\frac{1}{2}})$ as $z\to 1$ and $0(z^{-2})$ as $z\to\infty$, application of Cauchy's theorem shows that $I=0$.

Hence only AP_2 contributes to $[W_L]_{\mathrm{Av}}$ and from (29) we find

$$(41)\quad \left\{\begin{aligned}
[W_L]_{\mathrm{Av}}&=-\tfrac{1}{2}k\operatorname{Re}\,[i\,A\int_{-1}^{1}(P_2^+ - P_2^-)\,\overline{f_1}\,dx]=\\
&=-\frac{k}{\pi}\operatorname{Im}\,[\Gamma_{10}\int_{-1}^{1}[x+(1-x)C(k)]\,(1-x^2)^{-\frac{1}{2}}\,\overline{f_1}(x)\,dx].
\end{aligned}\right.$$

Since the drag is a second-order quantity and the horizontal component of the velocity of the aerofoil relatively to the fluid is, in zero-order approximation, equal to -1, we have (in second-order approximation)

$$(42)\qquad [W_D]_{\mathrm{Av}}=-[D]_{\mathrm{Av}},$$

where $[D]_{\mathrm{Av}}$ is given by (40).

[12]) This is more than just showing rigorously that $[W_L]_{\mathrm{Av}}+[W_D]_{\mathrm{Av}}+[W_W]_{\mathrm{Av}}=0$. The *separate* evaluation of $[W_L]_{\mathrm{Av}}$ and $[W_D]_{\mathrm{Av}}$ (which is more difficult than the evaluation of the sum of these quantities) provides a check on the correctness of (40).

The average rate of kinetic energy that is carried off in the wake is (since the horizontal component of the velocity of the fluid is $+1$ in zero-order approximation),

$$[W_W]_{\mathrm{Av}} = \tfrac{1}{2} \lim_{x \to \infty} \int_{-\infty}^{\infty} \tfrac{1}{2} \,|\operatorname{grad} \varphi_1|^2 \, dy.$$

From (37) we have

$$|\operatorname{grad} \varphi_1|^2 = \tfrac{1}{2} k^2 \,|A|^2 \, e^{-2k|y|} + 0((x^2+y^2)^{-1})$$

as $x \to \infty$, uniformly in y. Hence

$$(43) \quad \begin{cases} [W_W]_{\mathrm{Av}} = \frac{1}{8} k^2 |A|^2 \int\limits_{-\infty}^{\infty} e^{-2k|y|} \, dy = \frac{1}{8} k \,|A|^2 = \\ = \frac{2}{\pi^2 k} |\Gamma_{10}|^2 \,|H_1^{(2)}(k) + i H_0^{(2)}(k)|^{-2} = \\ = \frac{1}{2\pi} [C(k) + \overline{C(k)} - 2\, C(k)\, \overline{C(k)}] \,|\Gamma_{10}|^2, \end{cases}$$

since, in virtue of $\overline{H_n^{(2)}(k)} = H_n^{(1)}(k)$ $(n = 0, 1)$ and a Wronskian relation, we have

$$C(k) + \overline{C(k)} - 2C(k)\,\overline{C(k)} = i \,\frac{H_0^{(2)}(k)\, H_1^{(2)}(k) - H_1^{(2)}(k)\, H_0^{(2)}(k)}{|H_1^{(2)}(k) + i\, H_0^{(2)}(k)|^2} = $$
$$= \frac{4}{\pi k} \,|H_1^{(2)}(k) + i\, H_0^{(2)}(k)|^{-2}.$$

Comparing (41), (42) and (43), we find

$$[W_L]_{\mathrm{Av}} + [W_D]_{\mathrm{Av}} + [W_W]_{\mathrm{Av}} = 0,$$

which constitutes the conjectured balance of energy.

10. *The limiting case $k = 0$. Steady motion*

Nearly all of the foregoing results remain valid in the case of steady motion ($k = 0$). Since in this case $f_1(x)$ (and thus $V(x) = \frac{df_1}{dx}$) can be assumed to be real, we have $\Omega(\bar{z}) = -\overline{\Omega(z)}$ and (9) and (10) reduce to

$$\varphi(x, y) = \operatorname{Re}\,[\Omega(z)], \frac{\partial \varphi}{\partial x} - i \frac{\partial \varphi}{\partial y} = \Omega'(z).$$

Since $H_1^{(2)}(k) = \frac{2i}{\pi k} + 0(k \log k)$, $H_0^{(2)}(k) = 0(\log k)$, as $k \to 0$, we have

$$C(0) = 1.$$

Hence (28) reduces to

$$(44) \qquad A \frac{d\Omega_2}{dz} = \frac{\Gamma_{10}}{\pi i} (z^2 - 1)^{-\frac{1}{2}}, \text{ where } \Gamma_{10} = \int_{-1}^{1} (1+s)^{\frac{1}{2}} (1-s)^{-\frac{1}{2}} \frac{df_1}{ds} \, ds.$$

Combination of these formulae with (10) gives

$$\frac{d\Omega}{dz} = -\frac{1}{\pi i} (z+1)^{-\frac{1}{2}} (z+1)^{\frac{1}{2}} \int_{-1}^{1} (1+s)^{\frac{1}{2}} (1-s)^{-\frac{1}{2}} \frac{df_1}{ds} \frac{ds}{s-z},$$

which shows that u_1 and v_1 are continuous outside Λ_1.

The formulae for lift and moment reduce to the well-known formulae

$$L = -2 \int_{-1}^{1} (1+s)^{\frac{1}{2}} (1-s)^{-\frac{1}{2}} \frac{df_1}{ds} ds, \quad M = 2 \int_{-1}^{1} (1-s^2)^{\frac{1}{2}} \frac{df_1}{ds} ds.$$

The only phase where essential use is made of the assumption $k > 0$, is the discussion of the behaviour of $\Omega_2(z)$ at infinity (sec. 6). But from (44) we directly infer that for $k=0$ $\Omega_2(z)$ has a logarithmic singularity at infinity ($A\Omega_2$ now is the complex potential of the cyclic component of the motion, the circulation being $2\Gamma_{10}$).

Finally, we note that the drag as well as all of the terms of the balance of energy tend to zero as $k \to 0$.

11. *Some remarks on the conditions to be imposed on* $f_1(x)$

In order to be able to use the results of Muskhelishvili, we have assumed thus far that $V(x)$ and consequently $\frac{df_1}{dx}$ are Hölder-continuous. Now, the difficulties in proving results for sectionally holomorphic functions arise from two sources, viz.: the irregularities of the boundary functions and those of the boundary curves. In Muskhelishvili's work the accepted irregularities of the two types are of the same order: both the functions and the curves are assumed to satisfy Hölder-conditions. In the present problem, however, the boundary curves are straight lines and consequently the conditions on $\frac{df_1}{dx}$ can be weakened considerably, provided that the continuity conditions on φ_1 are equally relaxed. In view of results of Nickel [7], it seems to be sufficient to assume that for some $p > 1$

$$(1-x^2)^{-\frac{1}{2}} \left| \frac{df_1}{dx} \cdot (1-x^2)^{\frac{1}{2}} \right|^p$$

is (Lebesgue) integrable on $(-1, 1)$, all results then being interpreted in the "nearly everywhere" sense. However, in order to be able to make asymptotic estimates near $z = \pm 1$, a local Hölder condition seems to be indispensable (cf. Tricomi, l.c., p. 209).

From a physical point of view, sufficient generality is achieved if we suppose that $\frac{df_1}{dx}$ is Hölder continuous except at a finite number of points $c_1, \ldots, c_n$ of $-1 \leqslant x \leqslant 1$ and that near these points $\frac{df}{dx} = 0(|x-c_j|^{-\frac{1}{2}+\varepsilon})$ with $\varepsilon > 0$. Then the only thing that has to be changed is that grad φ_1 should be allowed to have singularities of the same character. Especially, if $x=1$ is a point of discontinuity, the Kutta condition should be weakened into grad $\varphi_1 = 0(|z-1|^{-\frac{1}{2}+\varepsilon})$. Of course, the conditions justifying the linearization are violated in the points of discontinuity. But since $|\text{grad}\,\varphi|^2$ remains integrable, the expressions for lift, moment, drag, etc., certainly will remain correct (in the appropriate orders of approximation).

Mathematical Institute,
University of Utrecht

REFERENCES

1. HEEMERT, A. VAN and J. H. GREIDANUS, Theory of the oscillating aerofoil in two-dimensional incompressible flow, Report F **41**, Nat. Aeron. Res. Inst., Amsterdam (1948).
2. REISSNER, E., Boundary value problems in aerodynamics of lifting surfaces in non-uniform motion, Bull. Am. Math. Soc., **55**, 825–850 (1949).
3. TIMMAN, R., La théorie des profils minces en écoulement non stationnaire en fluide incompressible ou compressible, Publ. Sci. et Techn. du Min. de l'Air, No. **296**, (1952).
4. SCHWARZ, L., Berechnung der Druckverteilung einer harmonisch sich verformenden Tragfläche in ebener Strömung, Luftfahrtforschung, **17**, 379–386 (1940).
5. KÜSSNER, H. G. and L. SCHWARZ, Der schwingende Flügel mit aerodynamisch ausgeglichenem Ruder, Luftfahrtforschung, **17**, 337–354 (1940).
6. NICKEL, K., Lösung eines speziellen Minimumproblems, Math. Zeitschr., **53**, 21–52 (1950).
7. ————, Lösung eines Integralgleichungssystems aus der Tragflügeltheorie, Math. Zeitschr., **54**, 81–96 (1951).
8. SÖHNGEN, H., Die Lösungen der Integralgleichung $g(x) = (1/2\pi) \oint f(\xi)\, d\xi/(x - \xi)$ und deren Anwendungen in der Tragflügeltheorie, Math. Zeitschr., **45**, 245–264 (1939).
9. TRICOMI, F. G., On the finite Hilbert transformation, Quart. J. Math., Oxf. Ser., (2), **2**, 199–211 (1951).
10. JONES, D. S., The unsteady motion of a thin aerofoil in an incompressible fluid, Comm. Pure and Appl. Math., **10**, 1–21 (1957).
11. MUSKHELISHVILI, N. I., Singular integral equations (english transl.), Groningen (1953).
12. ERDÉLYI, c.s., Higher transcendental functions, Vol. **II**, N.Y. (1953).
13. THEODORSON, T., General theory of aerodynamical instability and the mechanism of flutter, NACA T.R. No. **496** (1934).
14. LIGHTHILL, M. J., A new approach to thin aerofoil theory, Aeron. Quart., **3**, 193–210 (1951).